Exam Numerieke Wiskunde 1
June 19, 2013

Use of a simple calculator is allowed. All answers need to be motivated.
In front of the exercises you find its weight. In fact it gives the number of tenths which can be gained in the final mark. In total 5.1 points can be scored with this exam.

1. (a) $\boxed{4}$ Describe the Jacobi iteration method for solving $Ax = b$. Suppose that A is strictly diagonally dominant. Show that the Jacobi method converges for such matrices.

   (b) $\boxed{4}$ For an important class of matrices the Jacobi iteration matrix has real eigenvalues and $\lambda_2 = -\lambda_1$. Moreover $|\lambda_1| = |\lambda_2| > |\lambda_3| \geq \cdots \geq |\lambda_n|$. Show that eventually the error in the iteration is in the linear subspace spanned by the eigenvectors related to λ_1 and λ_2 and that the precise combination of the two depends on the error at the start of the iteration.

   (c) $\boxed{2}$ Explain why in linearly converging methods for $Ax = b$ the stopping criterion $||x^{(m+1)} - x^{(m)}||$ is small does not necessarily mean that $||x^{(m+1)} - x||$ is small, where x is the exact solution of $Ax = b$. When in particular is this the case?

2. Suppose $f_1 = x_1 + x_2 - 1, \quad f_2 = \frac{1}{100} - \ln(1 + x_2 - x_1)$. We want to solve $f_1(x_1, x_2) = 0, \quad f_2(x_1, x_2) = 0$, which is written in vector form as $\mathbf{f}(\mathbf{x}) = \mathbf{0}$. We want to solve it with a fixed point method $\mathbf{x_{n+1}} = \mathbf{g}(\mathbf{x_n})$ where $\mathbf{g}(\mathbf{x}) = \mathbf{x} + A\mathbf{f}(\mathbf{x})$.

   (a) $\boxed{3}$ Give the Jacobian matrix of $\mathbf{f}$.

   (b) $\boxed{2}$ Why is $(1/2, 1/2)$ a reasonable guess of the zero?

   (c) $\boxed{4}$ Derive the equation from which the best matrix A, based on the guess in the previous part, must be solved.

   (d) $\boxed{3}$ The Jacobian of $\mathbf{g}$ at the fixed point using the A of the previous part is

$$0.00502508 \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}$$

   Why is this matrix relevant for the study of the convergence of the fixed point method? Will the method converge in the neighborhood of the fixed point and why?

Continue on other side!

3. (a) [4] Give the general form of the interpolation error when $(n+1)$ interpolation points are used. Determine with this an upperbound for the interpolation error on the interval [0,2] for the interpolation of $f(x) = x^2 + 100x + 55$ using the interpolation points 0, 1 and 2 (given is that $|x(x-1)(x-2)| \leq \frac{2}{9}\sqrt{3}$ on [0,2]).

   (b) [3] Give the midpoint rule for integration on the interval $[a, b]$. Give also the interpolating polynomial on which this rule is based.

   (c) [4] The trapezium rule is based on two interpolation points and is exact for linear polynomials. However, the midpoint rule is also exact for linear polynomials. Explain from a sketch why this happens.

   (d) [5] Suppose we have a numerical method which approximates I by $I(h)$, where h is the mesh size and $I = I(h) + ch^4 + O(h^5)$ for some nonzero c. Derive a combination of $I(h)$ and $I(2h)$ that approximates I and which has error $O(h^5)$.

4. Consider on [0, 1] for $u(x,t)$ the convection equation $\partial u/\partial t = -\partial u/\partial x$ with initial condition $u(x,0) = \sin(\pi x)$ and boundary condition $u(0,t) = \sin^2(t)$. Let the grid in x direction be given by $x_i = i\Delta x$ where $\Delta x = 1/m$.

   (a) [3] Show that $u_x(x_i,t) = \frac{u(x_i,t)-u(x_{i-1},t)}{\Delta x} + O(\Delta x)$.

   (b) [4] Show that the system of ordinary differential equations (ODEs) that results from using the expression in (a) is of the form

$$\frac{d}{dt}\vec{u}(t) = -\frac{1}{\Delta x}(I - B)\vec{u}(t) + \vec{b}(t)$$

   and give B and $\vec{b}(t)$.

   (c) [3] Show that any eigenvalue of B will be at most one in magnitude. Sketch in the complex plane where the eigenvalues of $I - B$ are located.

   (d) [3] Derive the region of stability of the Forward/Explicit Euler method. Use this to show that the numerical integration with the Forward/Explicit Euler method of the system of ODEs will be stable if $\Delta t/\Delta x \leq 1$.

Total [51]